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Involutive orbits of non-Noether symmetry groups

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Abstract
We consider a set of functions on the Poisson manifold related by a continuous one-parameter group of transformations. A class of vector fields that produce involutive families of functions is investigated and the relationship between these vector fields and non-Noether symmetries of Hamiltonian dynamical systems is outlined. The theory is illustrated with two examples: a modified Boussinesq system and a Broer–Kaup system.

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Involutive orbits

In Hamiltonian integrable models, conservation laws often form an involutive orbit of the one-parameter symmetry group. Such a symmetry carries important information about the integrable model and its bi-Hamiltonian structure. This paper is an attempt to describe a class of one-parameter groups of transformations of the Poisson manifold that possess involutive orbits and may be related to Hamiltonian integrable systems [3, 8, 11].

Let $C^\infty(M)$ be an algebra of smooth functions on a manifold M equipped with Poisson bracket

$$\{f, g\} = W(\mathrm{d}f \wedge \mathrm{d}g) \tag{1}$$

where W is a Poisson bivector field. Each vector field E on the manifold M gives rise to the one-parameter group of transformations of $C^\infty(M)$ algebra

$$g_z = \mathrm{e}^{zL_E} \tag{2}$$

where L_E denotes the Lie derivative along the vector field E. To any smooth function $J \in C^\infty(M)$ this group assigns an orbit that goes through J

$$J(z) = g_z(J) = \mathrm{e}^{zL_E}(J) = J + zL_E J + \tfrac{1}{2}z^2(L_E)^2 J + \cdots \tag{3}$$

the orbit $J(z)$ is called involutive if

$$\{J(x), J(y)\} = 0 \qquad \forall x, y \in \mathbb{R} \tag{4}$$

(throughout this paper $\mathbb{R}$ denotes the set of real numbers, while $\mathbb{N}$ stands for positive integers). Involutive orbits are often related to the integrable models where $J(z)$ plays the role of the involutive family of conservation laws.

Involutivity of orbit $J(z)$ depends on the nature of the vector field E and the function $J = J(0)$ and in general it is hard to describe all pairs (E, J) that produce the involutive orbits. However one interesting class of involutive orbits can be outlined by the following theorem:

Theorem 1. *For any non-Poisson* $(L_E W \neq 0)$ *vector field E satisfying property*

$$L_E^2 W = 0 \tag{5}$$

and any function J *such that*

$$W(\mathrm{d}L_E J) = cL_E(W)(\mathrm{d}J) \qquad c \in \mathbb{R} \setminus (0 \cup \mathbb{N}) \tag{6}$$

one-parameter family of functions $J(z) = \mathrm{e}^{zL_E}(J)$ *is involutive.*

Proof. By taking the Lie derivative of property (6) along the vector field E we get

$$L_E(W)(\mathrm{d}L_E J) + W(\mathrm{d}(L_E)^2 J) = cL_E^2(W)(\mathrm{d}J) + cL_E(W)(\mathrm{d}L_E J) \tag{7}$$

where c is a real constant which is neither zero nor a positive integer. Taking into account (5) one can rewrite the result as follows:

$$W(\mathrm{d}(L_E)^2 J) = (c-1)L_E(W)(\mathrm{d}L_E J) \tag{8}$$

that after m iterations produces

$$W(\mathrm{d}(L_E)^{m+1} J) = (c-m)L_E(W)(\mathrm{d}(L_E)^m J). \tag{9}$$

Now using this formula let us prove that the functions $J^{(m)} = (L_E)^m J$ are in involution. Indeed

$$\{J^{(k)}, J^{(m)}\} = W(\mathrm{d}J^{(k)} \wedge \mathrm{d}J^{(m)}). \tag{10}$$

Assuming that $k > m$ let us rewrite the Poisson bracket as follows:

$$\begin{aligned}
W(\mathrm{d}J^{(k)} \wedge \mathrm{d}J^{(m)}) &= W(\mathrm{d}(L_E)^k J \wedge \mathrm{d}J^{(m)}) = L_{W(\mathrm{d}(L_E)^k J)} J^{(m)} \\
&= (c-k+1)L_{L_E(W)(\mathrm{d}(L_E)^{k-1} J)} J^{(m)} \\
&= (c-k+1)L_E(W)(\mathrm{d}J^{(k-1)} \wedge \mathrm{d}J^{(m)}) \\
&= -(c-k+1)L_{L_E(W)(\mathrm{d}(L_E)^m J)} J^{(k-1)} \\
&= -\frac{c-k+1}{c-m} L_{W(\mathrm{d}(L_E)^{m+1} J)} J^{(k-1)} \\
&= \frac{c-k+1}{c-m} W(\mathrm{d}J^{(k-1)} \wedge \mathrm{d}J^{(m+1)}).
\end{aligned} \tag{11}$$

Thus we have

$$(c-m)\{J^{(k)}, J^{(m)}\} = (c-k+1)\{J^{(k-1)}, J^{(m+1)}\}. \tag{12}$$

Using this formula $2(m-k)$ times produces

$$\{J^{(k)}, J^{(m)}\} = \{J^{(m)}, J^{(k)}\} \tag{13}$$

and since the Poisson bracket is skew-symmetric we finally get

$$\{J^{(k)}, J^{(m)}\} = 0. \tag{14}$$

Thus functions $J^{(m)} = (L_E)^m J$ are in involution. At the same time the orbit $J(z)$ is a linear combination of the functions $J^{(m)}$ and thus it is involutive as well. □

Remark 1. Note that if $L_E^2(W) = 0$ then $L_E(W)$ is a Poisson bivector field compatible with W (see [1, 4, 5, 12, 13]). Moreover [12], for a given Poisson bivector field W the bivector field $L_E W$ is Poisson (and automatically compatible with W) if and only if the Schouten bracket of $L_E^2(W)$ and W vanishes.

Remark 2. Formula (9) implies that the vector field

$$S = (c-m)E + t(c-m+1)W(\mathrm{d}J^{(m+1)}) \tag{15}$$

is of non-Noether symmetry [2, 9] of the Hamiltonian dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \{J^{(m)}, f\}. \tag{16}$$

In other words non-Poisson vector field S commutes with time evolution defined by the Hamiltonian vector field

$$X = \frac{\partial}{\partial t} + W(\mathrm{d}J^{(m)}). \tag{17}$$

This fact can be checked directly

$$\begin{aligned}
[S, X] &= (c-m)[E, X] + t(c-m+1)[W(\mathrm{d}J^{(m+1)}), W(\mathrm{d}J^{(m)})] - (c-m+1)W(\mathrm{d}J^{(m+1)}) \\
&= (c-m)L_E(W)(\mathrm{d}J^{(m)}) + (c-m)W(\mathrm{d}L_E J^{(m)}) \\
&\quad + t(c-m+1)W(\mathrm{d}\{J^{(m+1)}, J^{(m)}\}) - (c-m+1)W(\mathrm{d}J^{(m+1)}) \\
&= W(\mathrm{d}J^{(m+1)}) + (c-m)W(\mathrm{d}J^{(m+1)}) - (c-m+1)W(\mathrm{d}J^{(m+1)}) = 0.
\end{aligned} \tag{18}$$

In a similar manner one can show that

$$[X, [X, E]] = 0 \tag{19}$$

and thus E is a master symmetry [10] of the Hamiltonian system (16). Note also that formula (9) means that functions $J^{(m)} = (L_E)^m J$ form a Lenard scheme with respect to the bi-Hamiltonian structure formed by Poisson bivector fields W and $L_E W$.

In many infinite-dimensional integrable Hamiltonian systems the Poisson bivector has nontrivial kernel, and a set of conservation laws belongs to the orbit of non-Noether symmetry group that goes through the centre of the Poisson algebra. This fact is reflected in the following theorem (the theorem follows from the result of example 3.1.7 of [4], in this paper an alternative proof is suggested):

Theorem 2. *If non-Poisson vector field E satisfies property*

$$L_E^2 W = 0 \tag{20}$$

then every orbit derived from centre I of Poisson algebra $C^\infty(M)$ is involutive.

Proof. If function J belongs to centre $J \in I$ of Poisson algebra $C^\infty(M)$ then by definition $W(\mathrm{d}J) = 0$. By taking the Lie derivative of this condition along vector field E one gets

$$W(\mathrm{d}L_E J) = -L_E(W)(\mathrm{d}J) \tag{21}$$

that according to theorem 1 ensures involutivity of the $J(z)$ orbit. □

Modified Boussinesq system

The theorems stated above may have an interesting applications in the theory of infinite-dimensional Hamiltonian models where they provide a simple way to construct involutive families of conservation laws. One non-trivial example of such a model is a modified Boussinesq system [6, 14, 15] described by the following set of partial differential equations:

$$u_t = cv_{xx} + u_x v + uv_x \qquad v_t = -cu_{xx} + uu_x + 3vv_x \tag{22}$$

where $u = u(x, t)$, $v = v(x, t)$ are smooth functions on $\mathbb{R}^2$ subjected to the zero boundary conditions

$$\lim_{x\to\infty} x^m D_x^r u(x, t) = \lim_{x\to\infty} x^m D_x^r v(x, t) = 0 \qquad \forall m, r \in \mathbb{R}. \tag{23}$$

This system can be rewritten in the Hamiltonian form

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \{h, f\} = W(\mathrm{d}h \wedge \mathrm{d}f) \tag{24}$$

with the following Hamiltonian:

$$h = \frac{1}{2}\int_{-\infty}^{+\infty} (u^2 v + v^3 + 2cuv_x)\,\mathrm{d}x \tag{25}$$

and a Poisson bracket defined for any smooth functionals $\mathcal{K}$, $\mathcal{L}$ by

$$\{\mathcal{K}, \mathcal{L}\}_W = W(\mathrm{d}\mathcal{K} \wedge \mathrm{d}\mathcal{L}) = \int_{-\infty}^{+\infty} \left(\frac{\delta\mathcal{K}}{\delta u} D_x \left(\frac{\delta\mathcal{L}}{\delta u}\right) + \frac{\delta\mathcal{K}}{\delta v} D_x \left(\frac{\delta\mathcal{L}}{\delta v}\right)\right) \mathrm{d}x \tag{26}$$

where $\frac{\delta}{\delta u}$ denotes a variational derivative with respect to u. For Poisson bivector defined by (26) there exists a vector field E such that

$$L_E^2 W = 0. \tag{27}$$

The vector field has the following form:

$$E = -(uv + 2cv_x + x((uv)_x + cv_{xx}))\frac{\partial}{\partial u} - (u^2 + 2v^2 - 2cu_x + x(uu_x + 3vv_x - cu_{xx}))\frac{\partial}{\partial v}. \tag{28}$$

Applying the one-parameter group of transformations generated by this vector field to the centre of the Poisson algebra which in our case is formed by the functional

$$J = \int_{-\infty}^{+\infty} (ku + mv)\,\mathrm{d}x \tag{29}$$

where k, m are arbitrary constants, produces an involutive orbit that recovers the infinite sequence of conservation laws of the modified Boussinesq hierarchy

$$\begin{aligned}
J^{(0)} &= \int_{-\infty}^{+\infty} (ku + mv)\,\mathrm{d}x \\
J^{(1)} &= L_E J^{(0)} = \frac{m}{2}\int_{-\infty}^{+\infty} (u^2 + v^2)\,\mathrm{d}x \\
J^{(2)} &= (L_E)^2 J^{(0)} = m\int_{-\infty}^{+\infty} (u^2 v + v^3 + 2cuv_x)\,\mathrm{d}x \\
J^{(3)} &= (L_E)^3 J^{(0)} = \frac{3m}{4}\int_{-\infty}^{+\infty} \left(u^4 + 5v^4 + 6u^2v^2 - 12cv^2u_x + 4c^2u_x^2 + 4c^2v_x^2\right)\mathrm{d}x \\
J^{(r)} &= (L_E)^r J^{(0)} = L_E J^{(r-1)}.
\end{aligned} \tag{30}$$

Broer–Kaup system

Another interesting model that has an infinite sequence of conservation laws lying on a single orbit of the non-Noether symmetry group is the Broer–Kaup system [7, 14, 15] or more precisely a special case of the Broer–Kaup system formed by the following partial differential equations:

$$u_t = cu_{xx} + 2uu_x \qquad v_t = -cv_{xx} + 2uv_x + 2u_x v \tag{31}$$

where $u = u(x,t)$, $v = v(x,t)$ are again smooth functions on $\mathbb{R}^2$ subject to the zero boundary conditions

$$\lim_{x\to\infty} x^m D_x^r u(x,t) = \lim_{x\to\infty} x^m D_x^r v(x,t) = 0 \qquad \forall m, r \in \mathbb{R}. \tag{32}$$

Equations (31) can be rewritten in the Hamiltonian form

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \{h, f\} = W(\mathrm{d}h \wedge \mathrm{d}f) \tag{33}$$

with the Hamiltonian equal to

$$h = \int_{-\infty}^{+\infty} (u^2 v + cu_x v)\,\mathrm{d}x \tag{34}$$

and the Poisson bracket defined by

$$\{\mathcal{K}, \mathcal{L}\}_W = W(\mathrm{d}\mathcal{K} \wedge \mathrm{d}\mathcal{L}) = \int_{-\infty}^{+\infty} \left(\frac{\delta\mathcal{K}}{\delta u} D_x \left(\frac{\delta\mathcal{L}}{\delta v}\right) + \frac{\delta\mathcal{K}}{\delta v} D_x \left(\frac{\delta\mathcal{L}}{\delta u}\right)\right) \mathrm{d}x. \tag{35}$$

One can show that the following vector field E:

$$E = -(u^2 + 2cu_x + x(2uu_x + cu_{xx}))\frac{\partial}{\partial u} - (3uv - 2cv_x + x(2(uv)_x - cv_{xx}))\frac{\partial}{\partial v}, \tag{36}$$

has the property

$$L_E^2 W = 0 \tag{37}$$

and thus the group of transformations generated by this vector field transforms the centre of the Poisson algebra formed by the functionals

$$J = \int_{-\infty}^{+\infty} (ku + mv)\,\mathrm{d}x \tag{38}$$

into an involutive orbit that reproduces the well-known infinite set of conservation laws of the modified Broer–Kaup hierarchy

$$\begin{aligned}
J^{(0)} &= \int_{-\infty}^{+\infty} (ku + mv)\,\mathrm{d}x \\
J^{(1)} &= L_E J^{(0)} = m \int_{-\infty}^{+\infty} uv\,\mathrm{d}x \\
J^{(2)} &= (L_E)^2 J^{(0)} = 2m \int_{-\infty}^{+\infty} (u^2 v + cu_x v)\,\mathrm{d}x \\
J^{(3)} &= (L_E)^3 J^{(0)} = 3m \int_{-\infty}^{+\infty} (2u^3 v - 3cu^2 v_x - 2c^2 u_x v_x)\,\mathrm{d}x \\
J^{(r)} &= (L_E)^r J^{(0)} = L_E J^{(r-1)}.
\end{aligned} \tag{39}$$

Generalized modified Boussinesq system

The two examples discussed above are representatives of one interesting family of the infinite-dimensional Hamiltonian systems formed by D partial differential equations of the following type:

$$\begin{aligned}
&U_t = -2FGU_{xx} + \langle U, GU_x\rangle C + \langle C, GU_x\rangle U + \langle C, GU\rangle U_x \\
&\det G \neq 0, \qquad G^T = G, \qquad F^T = -F \\
&F_{mn}C_k + F_{km}C_n + F_{nk}C_m = 0
\end{aligned} \tag{40}$$

where U is the vector with the components u_m that are smooth functions on $\mathbb{R}^2$ subject to the zero boundary conditions

$$\lim_{x\to\infty} x^m D_x^r u_k(x,t) = 0 \qquad \forall m, r \in \mathbb{R}, k = 1, \ldots, D. \tag{41}$$

G is a constant symmetric nondegenerate matrix, F is a constant skew-symmetric matrix, C is a constant vector that satisfies the condition

$$F_{mn}C_k + F_{km}C_n + F_{nk}C_m = 0 \tag{42}$$

and $\langle , \rangle$ denotes the scalar product

$$\langle X, Y\rangle = \sum_{m=1}^{D} X_m Y_m. \tag{43}$$

The system of equations (40) is Hamiltonian with respect to the Poisson bivector defined by

$$\{\mathcal{K}, \mathcal{L}\}_W = W(\mathrm{d}\mathcal{K} \wedge \mathrm{d}\mathcal{L}) = \int_{-\infty}^{+\infty} \left\langle \frac{\delta\mathcal{K}}{\delta U}, G^{-1} D_x \left(\frac{\delta\mathcal{L}}{\delta U}\right)\right\rangle \mathrm{d}x \tag{44}$$

where $\frac{\delta}{\delta U}$ is a vector formed by the variational derivative $\frac{\delta}{\delta u_m}$. Moreover this model is actually bi-Hamiltonian as there exists another invariant Poisson bivector

$$\begin{aligned}
\{\mathcal{K}, \mathcal{L}\}_{\hat{W}} &= \hat{W}(\mathrm{d}\mathcal{K} \wedge \mathrm{d}\mathcal{L}) \\
&= \int_{-\infty}^{+\infty} \left(\left\langle C, \frac{\delta\mathcal{K}}{\delta U}\right\rangle \left\langle U, D_x\left(\frac{\delta\mathcal{L}}{\delta U}\right)\right\rangle + \left\langle U, \frac{\delta\mathcal{K}}{\delta U}\right\rangle \left\langle C, D_x\left(\frac{\delta\mathcal{L}}{\delta U}\right)\right\rangle \right. \\
&\quad \left. + \left\langle U_x, \frac{\delta\mathcal{K}}{\delta U}\right\rangle \left\langle C, \frac{\delta\mathcal{L}}{\delta U}\right\rangle - 2\left\langle \frac{\delta\mathcal{K}}{\delta U}, F D_x^2 \frac{\delta\mathcal{L}}{\delta U}\right\rangle \right) \mathrm{d}x
\end{aligned} \tag{45}$$

that is compatible with W. Corresponding Hamiltonians that produce bi-Hamiltonian realization

$$\frac{\mathrm{d}}{\mathrm{d}t} U = \hat{W}(\mathrm{d}\hat{H} \wedge \mathrm{d}U) = W(\mathrm{d}H \wedge \mathrm{d}U) \tag{46}$$

of the evolutionary equations (40) are

$$\hat{H} = \frac{1}{2}\int_{-\infty}^{+\infty} \langle U, GU\rangle \,\mathrm{d}x \tag{47}$$

and

$$H = \frac{1}{2}\int_{-\infty}^{+\infty} \{\langle C, GU\rangle\langle U, GU\rangle + 2\langle FGU_x, GU\rangle\} \,\mathrm{d}x. \tag{48}$$

The most remarkable property of system (40) is that it possesses a set of conservation laws that belong to the single orbit obtained from the centre of Poisson algebra via a one-parameter group of transformations generated by the following vector field:

$$\begin{aligned}
E = {}& \langle C, GU\rangle\langle U, \partial_U\rangle + \langle U, GU\rangle\langle C, \partial_U\rangle + 4\langle FGU_x, \partial_U\rangle + x(\langle C, GU_x\rangle\langle U, \partial_U\rangle \\
&+ \langle C, GU\rangle\langle U_x, \partial_U\rangle + \langle U, GU_x\rangle\langle C, \partial_U\rangle + 2\langle FGU_{xx}, \partial_U\rangle).
\end{aligned} \tag{49}$$

Note that the centre of Poisson algebra (with respect to the bracket defined by W) is formed by functionals of the following type:

$$J = \int_{-\infty}^{+\infty} \langle K, U\rangle \,\mathrm{d}x \tag{50}$$

where K is an arbitrary constant vector and applying the group of transformations generated by E to the functional J yields the infinite sequence of the functionals

$$\begin{aligned}
J^{(0)} &= \int_{-\infty}^{+\infty} \langle K, U\rangle \,\mathrm{d}x\\
J^{(1)} &= L_E J^{(0)} = \frac{1}{2}\langle C, K\rangle \int_{-\infty}^{+\infty} \langle U, GU\rangle \,\mathrm{d}x\\
J^{(2)} &= (L_E)^2 J^{(0)} = \langle C, K\rangle \int_{-\infty}^{+\infty} \{\langle C, GU\rangle\langle U, GU\rangle + 2\langle FGU_x, GU\rangle\}\,\mathrm{d}x\\
J^{(3)} &= (L_E)^3 J^{(0)} = \frac{1}{4}\langle C, K\rangle \int_{-\infty}^{+\infty} \{3\langle C, GC\rangle\langle U, GU\rangle^2 + 12\langle C, GU\rangle^2\langle U, GU\rangle + 32\langle C, GU\rangle\\
&\quad \times\langle GU, FGU_x\rangle + 24\langle U, GU\rangle\langle GC, FGU_x\rangle + 48\langle FGU_x, GFGU_x\rangle\}\,\mathrm{d}x\\
J^{(r)} &= (L_E)^r J^{(0)} = L_E J^{(r-1)}.
\end{aligned} \tag{51}$$

One can check that the vector field E satisfies the condition

$$L_E^2 W = 0 \tag{52}$$

and according to theorem 2 the sequence $J^{(m)}$ is involutive. So $J^{(m)}$ are conservation laws of bi-Hamiltonian dynamical system (40) and vector field E is related to non-Noether symmetries of evolutionary equations (see remark 2).

Note that in the special case when C, F, G, K have the following form:

$$D = 2, \qquad F_{12} = -F_{21} = \tfrac{1}{2}c, \qquad C = K = (0, 1), \qquad G = 1 \tag{53}$$

model (40) reduces to the modified Boussinesq system discussed above. Another choice of constants C, F, G, K

$$\begin{aligned}
&D = 2, \qquad F_{12} = -F_{21} = \tfrac{1}{2}c, \qquad C = K = (0, 1)\\
&G_{12} = G_{21} = 1, \qquad G_{11} = G_{22} = 0
\end{aligned} \tag{54}$$

gives rise to the Broer–Kaup system described in the previous example.

Conclusions

Groups of transformations of the Poisson manifold that possess an involutive orbit play an important role in some integrable models where the conservation laws form an orbit of non-Noether symmetry group. Therefore the classification of the vector fields that generate such a group would create a good background for the description of a remarkable class of integrable systems that have an interesting geometric origin. This paper is an attempt to outline one particular class of the vector fields that are related to non-Noether symmetries of Hamiltonian dynamical systems and produce an involutive family of conservation laws.

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